TENSOR PRODUCTS AND NUCLEARITY OF ORDERED VECTOR SPACES WITH ARCHIMEDEAN ORDER UNIT

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Abstract. We introduce the injective tensor products and the projective tensor products of ordered vector spaces with Archimedean order unit and study their functorial properties. The local characterization of a nuclear space is given.

1. Introduction and preliminary

Kadison proved that every ordered real vector space with Archimedean order unit can be embedded into a real continuous function algebra on a compact Hausdorff space via a unital order isomorphism [Ka].

A real vector space V is called an ordered real vector space if there exists a cone $V^+ \subset V$ such that $V^+ \cap -V^+ = \{0\}$. The cone V^+ induces a partial order by $v \geqslant w$ if and only if $v - w \in V^+$. For an ordered real vector space (V, V^+), an element e in V is called an order unit if for each v in V, there exists a real number $r > 0$ such that $re \geqslant v$. We call an order unit e Archimedean order unit if $\varepsilon e + v \in V^+$ for any $\varepsilon > 0$ implies $v \in V^+$. The order norm of an ordered real vector space with Archimedean order unit is defined by

$$\|v\| = \inf\{r > 0 : -re \leqslant v \leqslant re\}.$$

It is obvious that the unital subspace of a real continuous function algebra is an ordered real vector space with Archimedean order unit. Kadison's representation theorem tells the converse. In other words, the axioms of ordered real vector space with Archimedean order unit can be regarded as the abstract characterization of the unital subspace of a real continuous function algebra.

It is natural to consider the category consisting of ordered real vector spaces and unital positive maps. Unfortunately, this category misbehaves under the functorial operations, such as quotient and tensor product. This misbehavior can be remedied by the Archimedeanization process [PT].

For an ordered real vector space (V, V^+) with an order unit e, we let

$$D = \{v \in V : \varepsilon e + v \in V^+ \text{ for all } \varepsilon > 0\} \quad \textit{and} \quad N = D \cap -D = \bigcap_{f \in S(V)} \ker f,$$

where $S(V)$ denotes the state space on V. The Archimedeanization V_{Arch} of V is defined as an ordered real vector space $(V/N, D + N)$ with an order unit $e + N$. Then V_{Arch} is an ordered real vector space with the Archimedean order unit. It has the universal property: for an ordered real vector space W with Archimedean order unit and a unital positive map $\varphi : V \to W$, there exists a unique positive linear map $\tilde{\varphi} : V_{\mathrm{Arch}} \to W$ with

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$\varphi = \tilde{\phi} \circ q$.

$$\begin{tikzcd} V \arrow[rr, "q"] \arrow[dr, "\varphi"'] & & V_{\mathrm{Arch}} \arrow[dl, "\tilde{\varphi}"] \\ & W & \end{tikzcd}$$

We say that a subspace J of V is an order ideal of V if $p \in J$ and $0 \leqslant q \leqslant p$ imply that $q \in J$. The Archimedean quotient of V by J is defined as the Archimedeanization of $(V/J, V^+ + J)$ with Archimedean order unit $e + J$. For a unital positive linear map $\varphi : V \to W$, the Archimedean quotient by $\ker\varphi$ is unitally order isomorphic to $V/\ker\varphi$ with positive cone

$$(V/\ker\varphi)^+ = \{v + \ker\varphi : \forall\varepsilon > 0, \exists j \in \ker\varphi \text{ such that } j + \varepsilon e + v \in V^+\}$$

and Archimedean order unit $e + \ker\varphi$. The map $\tilde{\varphi} : V/\ker\varphi \to W$ given by $\tilde{\varphi}(v + \ker\varphi) = \varphi(v)$ is a unital positive linear map.

$$\begin{tikzcd} V \arrow[rr, "\varphi"] \arrow[dr, "q"'] & & W \\ & V/\ker\varphi \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

The universal property and the first isomorphism theorem justify the Archimedeanization.

In section 2, we introduce the injective tensor products and the projective tensor products of ordered real vector spaces with Archimedean order unit. We prove that they are also ordered real vector spaces with Archimedean order unit and the tensor product of unital positive maps is also a unital positive map in each case. It is proved that the injective tensor product is injective and the projective tensor product is projective.

We call an ordered real vector space with Archimedean order unit nuclear if the injective tensor product with any other one coincides with the projective tensor product. In section 3, we give the local characterization of a nuclear space: an ordered real vector space V with Archimedean order unit is nuclear if and only if there exist nets of unital positive maps $\Phi_\lambda : V \to \ell^\infty_{n_\lambda}$ and $\Psi_\lambda : \ell^\infty_{n_\lambda} \to V$ such that $\Psi_\lambda \circ \Phi_\lambda$ converges to id_V in the point-norm topology.

A $*$-vector space consists of a complex vector space V together with involution. We denote $V_h = \{x \in V : x^* = x\}$. A $*$-vector space V is called an ordered $*$-vector space if there exists a cone $V^+ \subset V_h$ such that $V^+ \cap -V^+ = \{0\}$. The cone V^+ induces a partial order on V_h by $v \geqslant w$ if and only if $v - w \in V^+$. For an ordered $*$-vector space (V, V^+), an element e in V is called an order unit if for each v in V_h, there exists a real number $r > 0$ such that $re \geqslant v$. We say that an order unit e is Archimedean order unit if $\varepsilon e + v \in V^+$ for any $\varepsilon > 0$ implies $v \in V^+$.

In contrary to the case of real ordered vector space with Archimedean order unit, the order structure does not determine the norm structure in unique way. The order structure determines only the norms of hermitian elements. A norm $\|\cdot\|$ is called $*$-norm if $\|v^*\| = \|v\|$ for all $v \in V$. A $*$-norm is called an order norm if it extends the order norm on V_h. The minimal order norm $\|\cdot\|_m : V \to [0,\infty)$ is defined by

$$\|v\|_m = \sup\{|f(v)| : f : V \to \mathbb{C} \text{ is a state}\}.$$

And the maximal order norm $\|\cdot\|_M : V \to [0,\infty)$ is defined by

$$\|v\|_M = \inf\{\sum_{i=1}^n |\lambda_i|\|v_i\| : v = \sum_{i=1}^n \lambda_i v_i \text{ with } v_i \in V_h \text{ and } \lambda_i \in \mathbb{C}\}.$$

The minimal order norm and the maximal order norm are order norms. If $\|\cdot\|$ is an order norm, then we have $\|v\|_m \leqslant \|v\| \leqslant \|v\|_M$ for all $v \in V$.

For an ordered $*$-vector space (V, V^+) with an order unit e, we let

$$D = \{v \in V_h : \varepsilon e + v \in V^+ \text{ for all } \varepsilon > 0\} \quad \text{and} \quad N = \bigcap_{f \in S(V)} \ker f.$$

The Archimedeanization V_{Arch} of V is defined as an ordered $*$-vector space $(V/N, D+N)$ with an order unit $e+N$. Then V_{Arch} is an ordered $*$-vector space with Archimedean order unit and it has the universal property.

We say that a self-adjoint subspace J of V is an order ideal of V if $p \in J$ and $0 \leqslant q \leqslant p$ imply that $q \in J$. The Archimedean quotient of V by J is defined as the Archimedeanization of $(V/J, V^+ + J)$ with order unit $e + J$. For a unital positive linear map $\varphi : V \to W$ between ordered $*$-vector spaces with Archimedean order unit, the Archimedean quotient by $\ker\varphi$ is unitally order isomorphic to $V/\ker\varphi$ with positive cone

$$(V/\ker\varphi)^+ = \{v + \ker\varphi : \forall \varepsilon > 0, \exists j \in \ker\varphi \text{ such that } j + \varepsilon e + v \in V^+\}$$

and Archimedean order unit $e + \ker\varphi$. The map $\tilde{\varphi} : V/\ker\varphi \to W$ given by $\tilde{\varphi}(v + \ker\varphi) = \varphi(v)$ is a unital positive linear map.

In section 4, we obtain the results similar to the previous sections for the case of ordered $*$-vector spaces with Archimedean order unit.

The key reference of this paper is [PT]. It contains the detailed expositions on the preliminaries introduced in this section. For brevity, we call the ordered real vector space with Archimedean order unit as real AOU space and we call the ordered $*$-vector space with Archimedean order unit as AOU space from now on.

2. Tensor products of real AOU spaces

In this section, we introduce the injective tensor products and the projective tensor products of real AOU spaces and study their functorial properties. Our model is Grothendieck's tensor theory [DF, G]. For real AOU spaces V and W, we denote by $S(V)$ the state space on V and denote by $V^+ \otimes W^+$ the cone $\{\sum_i v_i \otimes w_i \in V \otimes W : v_i \in V^+, w_i \in W^+\}$.

Definition 2.1. Suppose that (V, V^+, e_V) and (W, W^+, e_W) are real AOU spaces.

(1) we define *an injective tensor product* $V \otimes_\varepsilon W$ as $(V \otimes W, (V \otimes_\varepsilon W)^+, e_V \otimes e_W)$ where $(V \otimes_\varepsilon W)^+ = \{z \in V \otimes W : (f \otimes g)(z) \geqslant 0 \text{ for all } f \in S(V), g \in S(W)\}$.

(2) we define *a projective tensor product* $V \otimes_\pi W$ as $(V \otimes W, (V \otimes_\pi W)^+, e_V \otimes e_W)$ where $(V \otimes_\pi W)^+ = \{z \in V \otimes W : z + \varepsilon e_V \otimes e_W \in V^+ \otimes W^+ \text{ for all } \varepsilon > 0\}$.

Theorem 2.2. *Suppose that (V, V^+, e_V) and (W, W^+, e_W) are real AOU spaces. Then*

(1) *the injective tensor product $V \otimes_\varepsilon W$ is a real AOU space.*

(2) *the projective tensor product $V \otimes_\pi W$ is a real AOU space.*

(3) *the order norms $\|\cdot\|_{V\otimes_\varepsilon W}$ and $\|\cdot\|_{V\otimes_\pi W}$ are cross norms with respect to the order norms of V and W. In addition, the inequality $\|\cdot\|_{V\otimes_\varepsilon W} \leqslant \|\cdot\|_{V\otimes_\pi W}$ holds.*

Proof. (1) It is obvious that $(V \otimes_\varepsilon W)^+$ is a cone. Let $z = \sum_{i=1}^n v_i \otimes w_i \in (V \otimes_\varepsilon W)^+ \cap -(V \otimes_\varepsilon W)^+$. We may assume that $\{v_i\}_{i=1}^n$ is linearly independent. For all states $f \in S(V)$ and $g \in S(W)$, we have

$$0 = (f \otimes g)(\sum_{i=1}^n v_i \otimes w_i) = \sum_{i=1}^n f(v_i)g(w_i) = f(\sum_{i=1}^n g(w_i)v_i).$$

By [PT, Prop 2.19], we have $\sum_{i=1}^n g(w_i)v_i = 0$. Since $\{v_i\}_{i=1}^n$ is linearly independent, we see that $g(w_i) = 0$ for all $g \in S(W)$. By [PT, Prop 2.19] again, we have $w_i = 0$ for all $1 \leqslant i \leqslant n$, thus $z = 0$.

For $v \in V, w \in W$ and $f \in S(V), g \in S(W)$, we have

$$(f \otimes g)(\|v\|\|w\|e_V \otimes e_W + v \otimes w) = \|v\|\|w\| + f(v)g(w) \geqslant 0,$$

thus $\|v\|\|w\| + v \otimes w \in (V \otimes_\varepsilon W)^+$.

Suppose that $z + \varepsilon e_V \otimes e_W \in (V \otimes_\varepsilon W)^+$ for any $\varepsilon > 0$. For all $f \in S(V)$ and $g \in S(W)$, we have

$$0 \leqslant (f \otimes g)(z + \varepsilon e_V \otimes e_W) = (f \otimes g)(z) + \varepsilon.$$

It follows that $(f \otimes g)(z) \geqslant 0$ for all $f \in S(V)$ and $g \in S(W)$, thus $z \in (V \otimes_\varepsilon W)^+$.

(2) From the inclusion $V^+ \otimes W^+ \subset (V \otimes_\varepsilon W)^+$, we see that $V^+ \otimes W^+ \cap -(V^+ \otimes W^+) = \{0\}$. For $v \in V$ and $w \in W$, we have

$$\begin{aligned}
&\|v\|\|w\|e_V \otimes e_W \pm v \otimes w \\
=&\frac{1}{2}(\|v\|e_V \pm v) \otimes (\|w\|e_W + w) + \frac{1}{2}(\|v\|e_V \mp v) \otimes (\|w\|e_W - w) \in V^+ \otimes W^+.
\end{aligned}$$

Hence, $(V \otimes W, V^+ \otimes W^+, e_V \otimes e_W)$ is an ordered real vector space with an order unit. For $f \in S(V)$ and $g \in S(W)$, $f \otimes g$ is a state on $(V \otimes W, V^+ \otimes W^+, e_V \otimes e_W)$. By the above proof (1), we have

$$N = \{z \in V \otimes W : F(z) = 0 \text{ for all states } F \text{ on } (V \otimes W, V^+ \otimes W^+, e_V \otimes e_W)\} = \{0\}.$$

Hence, the projective tensor product $V \otimes_\pi W$ is the Archimedeanization of $(V \otimes W, V^+ \otimes W^+, e_V \otimes e_W)$.

(3) Let $z \in (V \otimes_\pi W)^+$. For any $\varepsilon > 0$, we have

$$z + \varepsilon e_V \otimes e_W \in V^+ \otimes W^+ \subset (V \otimes_\varepsilon W)^+,$$

thus $z \in (V \otimes_\varepsilon W)^+$. From the inclusion $(V \otimes_\pi W)^+ \subset (V \otimes_\varepsilon W)^+$, we see that $\| \cdot \|_{V \otimes_\varepsilon W} \leqslant \| \cdot \|_{V \otimes_\pi W}$. By the above (2), it follows that

$$\begin{aligned}
\|v\|\|w\| &= \sup\{|(f \otimes g)(v \otimes w)| : f \in S(V), g(W)\} \\
&\leqslant \|v \otimes w\|_{V \otimes_\varepsilon W} \\
&\leqslant \|v \otimes w\|_{V \otimes_\pi W} \\
&\leqslant \|v\|\|w\|.
\end{aligned}$$

□

The projective tensor product of real AOU spaces is characterized by the following universal property.

Proposition 2.3. *Suppose that V, W and Z are real AOU spaces and $\Phi : V \times W \to Z$ is a bilinear map such that $\Phi(e_V, e_W) = e_Z$ and $\Phi(v, w) \in Z^+$ for all $v \in V^+$ and $w \in W^+$. Then there exists a unique unital positive linear map $\tilde{\Phi} : V \otimes_\pi W \to Z$ such that $\Phi(v, w) = \tilde{\Phi}(v \otimes w)$. This universal property characterizes the projective tensor product $V \otimes_\pi W$ up to unital order isomorphism.*

$$\begin{tikzcd} V \times W \arrow[rr, "\otimes"] \arrow[dr, "\Phi"'] & & V \otimes_\pi W \arrow[dl, "\tilde{\Phi}"] \\ & Z & \end{tikzcd}$$

Proof. Suppose that $z \in (V \otimes_\pi W)^+$. Then we have $z + \varepsilon e_V \otimes e_W \in V^+ \otimes_\pi W^+$ for any $\varepsilon > 0$. It follows that

$$\tilde{\Phi}(z) + \varepsilon e_Z = \tilde{\Phi}(z + \varepsilon e_V \otimes e_W) \in Z^+$$

for any $\varepsilon > 0$, thus $\tilde{\Phi}(z) \in Z^+$. □

Proposition 2.4. *Suppose that $S : V_1 \to V_2$ and $T : W_1 \to W_2$ are unital positive linear maps for real AOU spaces V_1, V_2, W_1, W_2. Then*

(1) *$S \otimes T : V_1 \otimes_\varepsilon W_1 \to V_2 \otimes_\varepsilon W_2$ is a unital positive linear map.*
(2) *$S \otimes T : V_1 \otimes_\pi W_1 \to V_2 \otimes_\pi W_2$ is a unital positive linear map.*

Proof. (1) Let $z = \sum_{i=1}^n v_i \otimes w_i \in (V_1 \otimes_\varepsilon W_1)^+$. For all $f \in S(V_2)$ and $g \in S(W_2)$, we have

$$(f \otimes g)(S \otimes T)(\sum_{i=1}^n v_i \otimes w_i) = \sum_{i=1}^n f \circ S(v_i)\ g \circ T(w_i) \geqslant 0$$

because $f \circ S \in S(V_1)$ and $g \circ T \in S(W_1)$. It follows that $(S \otimes T)(z) \in (V_2 \otimes_\varepsilon W_2)^+$.

(2) Let $z \in (V_1 \otimes_\pi W_1)^+$. Then we have $z + \varepsilon e_{V_1} \otimes e_{W_1} \in V_1^+ \otimes W_1^+$ for any $\varepsilon > 0$. It follows that

$$(S \otimes T)(z) + \varepsilon e_{V_2} \otimes e_{W_2} = (S \otimes T)(z + \varepsilon e_{V_1} \otimes e_{W_1}) \in V_2^+ \otimes W_2^+$$

for any $\varepsilon > 0$, thus $(S \otimes T)(z) \in (V_2 \otimes_\pi W_2)^+$. □

By virtue of Theorem 2.2 and Proposition 2.4, we can regard $\cdot \otimes_\varepsilon \cdot$ and $\cdot \otimes_\pi \cdot$ as the bifunctors from the category consisting of real AOU spaces and unital positive maps into itself.

Definition 2.5. Suppose that $T : V \to W$ is a unital positive surjective linear map for real AOU spaces V and W. We call $T : V \to W$ *an order quotient map* if for any w in W^+ and $\varepsilon > 0$, we can take an element v in V so that it satisfies

$$v + \varepsilon e_V \in V^+ \quad \text{and} \quad T(v) = w.$$

The key point of the above definition is that the lifting v depends on the choice of $\varepsilon > 0$. Slightly modifying [PT, Theorem 2.45], we get the following proposition. It justifies the terminology, order quotient map.

Proposition 2.6. *Suppose that $T : V \to W$ is a unital positive surjective linear map for real AOU spaces V and W. Then $T : V \to W$ is an order quotient map if and only if $\tilde{T} : V/\ker T \to W$ is an order isomorphism.*

Proof. $T : V \to W$ is an order quotient map

$\Leftrightarrow \forall w \in W^+, \forall \varepsilon > 0, \exists v \in V, v + \varepsilon e_V \in V^+$ and $T(v) = w$
$\Leftrightarrow \forall w \in W^+, \exists v \in V, v + \ker T \in (V/\ker T)^+$ and $\tilde{T}(v + \ker T) = w$
$\Leftrightarrow \tilde{T} : V/\ker T \to W$ is an order isomorphism.

□

Recall that a bounded linear map $T : V \to W$ for normed spaces V and W is called a quotient map if it maps the open unit ball of V onto the open unit ball of W.

Proposition 2.7. *Suppose that $T : V \to W$ is a unital positive linear map for real AOU spaces V and W. Then*

(1) *$T : V \to W$ is an order embedding if and only if it is an isometry.*
(2) *if $T : V \to W$ is a quotient map, then it is an order quotient map.*

Proof. (1) It follows from [PT, Prop 2.27] and [PT, Prop 2.28].

(2) Let $w \in W^+$ and $\varepsilon > 0$. Then we have

$$-\frac{1}{2}\|w\|e_W \leqslant w - \frac{1}{2}\|w\|e_W \leqslant \frac{1}{2}\|w\|e_W.$$

There exists an element v in V such that

$$q(v) = w - \frac{1}{2}\|w\|e_W \quad \text{and} \quad \|v\| \leqslant \frac{1}{2}\|w\| + \varepsilon.$$

It follows that

$$v + \frac{1}{2}\|w\|e_V + \varepsilon e_V \in V^+ \quad \text{and} \quad q(v + \frac{1}{2}\|w\|e_V) = w.$$

□

The following theorem justifies the terminologies, the injective tensor product and the projective tensor product.

Theorem 2.8. *(1) For real AOU spaces V_1, V_2, W and a unital order embedding $\iota : V_1 \to V_2$, the linear map $\iota \otimes id_W : V_1 \otimes_\varepsilon W \to V_2 \otimes_\varepsilon W$ is a unital order embedding.*

(2) For real AOU spaces V_1, V_2, W and an order quotient map $Q : V_1 \to V_2$, the linear map $Q \otimes id_W : V_1 \otimes_\pi W \to V_2 \otimes_\pi W$ is an order quotient map.

Proof. (1) Suppose that $(\iota \otimes id_W)(z) \in (V_2 \otimes_\varepsilon W_2)^+$ for $z \in V_1 \otimes W$. We can regard V_1 as a subspace of V_2. By Hahn-Banach type theorem [PT, Corollary 2.15], a state $f : V_1 \to \mathbb{R}$ extends to a state $\tilde{f} : V_2 \to \mathbb{R}$. For a state $g \in S(W)$, we have

$$0 \leqslant (\tilde{f} \otimes g)(\iota \otimes id_V)(z) = f \otimes g(z),$$

thus $z \in (V_1 \otimes_\varepsilon W)^+$.

(2) Let $z \in (V_2 \otimes_\pi W)^+$ and $\varepsilon > 0$. Then we have

$$z + \frac{\varepsilon}{2} e_{V_2} \otimes e_W \in V_2^+ \otimes W^+.$$

We write

$$z = \sum_{i=1}^n v_i \otimes w_i - \frac{\varepsilon}{2} e_{V_2} \otimes e_W$$

for $v_i \in V_2^+$ and $w_i \in W^+$. There exists u_i in V_1 such that

$$Q(u_i) = v_i \quad \text{and} \quad u_i + \frac{\varepsilon}{2n\|w_i\|} e_{V_1} \in V_1^+$$

for each $1 \leqslant i \leqslant n$. It follows that

$$(Q \otimes id_W)(\sum_{i=1}^n u_i \otimes w_i - \frac{\varepsilon}{2} e_{V_1} \otimes e_W) = \sum_{i=1}^n v_i \otimes w_i - \frac{\varepsilon}{2} e_{V_2} \otimes e_W = z$$

and

$$\begin{aligned}
&(\sum_{i=1}^n u_i \otimes w_i - \frac{\varepsilon}{2} e_{V_1} \otimes e_W) + \varepsilon e_{V_1} \otimes e_W \\
=&\sum_{i=1}^n (u_i + \frac{\varepsilon}{2n\|w_i\|} e_{V_1}) \otimes w_i + \sum_{i=1}^n \frac{\varepsilon}{2n} e_{V_1} \otimes (e_W - \frac{1}{\|w_i\|} w_i) \\
\in& V_1^+ \otimes W^+
\end{aligned}$$

□

We denote by $V \otimes_\lambda W$ the injective tensor product of normed spaces V and W.

Proposition 2.9. *For real AOU spaces V and W, the order norm on the injective tensor product $V \otimes_\varepsilon W$ coincides with the injective tensor norm on $V \otimes_\lambda W$.*

Proof. By Kadison's representation theorem [Ka], real AOU spaces V and W can be embedded into the real continuous function algebras $C(X)$ and $C(Y)$, respectively. By Theorem 2.8 and Proposition 2.7, $V \otimes_\varepsilon W$ is isometrically embedded into $C(X) \otimes_\varepsilon C(Y)$. Since every state on $C(X)$ is in the weak*-closed convex hull of Dirac measures on X, $C(X) \otimes_\varepsilon C(Y)$ is unital order isomorphically embedded into $C(X \times Y)$. Since the real continuous function algebra $C(X \times Y)$ is the completion of the tensor product $C(X) \otimes C(Y)$ with respect to the injective tensor norm λ, we see that $V \otimes_\varepsilon W \cong V \otimes_\lambda W$ isometrically. □

Let $\mathbb{M}_n(\mathbb{R})$ be $n \times n$ real matrix algebra. It is obvious that

$$\mathbb{M}_2^+(\mathbb{R}) \otimes_\pi \mathbb{M}_2^+(\mathbb{R}) \subset \mathbb{M}_4^+(\mathbb{R}) \subset \mathbb{M}_2^+(\mathbb{R}) \otimes_\varepsilon \mathbb{M}_2^+(\mathbb{R}).$$

Since the transpose map on $\mathbb{M}_2(\mathbb{R})$ is a unital positive map, we see that

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{M}_4^+(\mathbb{R}) \setminus \mathbb{M}_2^+(\mathbb{R}) \otimes_\pi \mathbb{M}_2^+(\mathbb{R}), \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{M}_2^+(\mathbb{R}) \otimes_\varepsilon \mathbb{M}_2^+(\mathbb{R}) \setminus \mathbb{M}_4^+(\mathbb{R}).$$

3. Local characterization of nuclear space

A C^*-algebra A is called nuclear if the identity $A \otimes_{\min} B = A \otimes_{\max} B$ holds for any C^*-algebra B. It is well known that a C^*-algebra A is nuclear if and only if there exist nets of u.c.p. maps $\Phi_\lambda : A \to \mathbb{M}_{n_\lambda}(\mathbb{C})$ and $\Psi_\lambda : \mathbb{M}_{n_\lambda}(\mathbb{C}) \to A$ such that $\Psi_\lambda \circ \Phi_\lambda : A \to A$ converges to id_A in the point-norm topology [CE, Ki].

Definition 3.1. A real AOU space V is called *nuclear* if the identity

$$V \otimes_\varepsilon W = V \otimes_\pi W$$

holds for any real AOU space W.

The purpose of this section is to prove that a real AOU space V is nuclear if and only if there exist nets of unital positive maps $\Phi_\lambda : V \to \ell^\infty_{n_\lambda}$ and $\Psi_\lambda : \ell^\infty_{n_\lambda} \to V$ such that $\Psi_\lambda \circ \Phi_\lambda$ converges to id_V in the point-norm topology.

Lemma 3.2. *For a real AOU space V, we set*

$$(V^* \oplus \mathbb{R})^+ = \{f + \lambda 1 \in V^* \oplus \mathbb{R} : f(v) + \lambda \geqslant 0 \text{ for all } 0 \leqslant v \leqslant e_V\}.$$

Then $(V^* \oplus \mathbb{R}, (V^* \oplus \mathbb{R})^+, 1)$ is a real AOU space.

Proof. It is obvious that $(V^* \oplus \mathbb{R})^+$ is a cone. Let $f + \lambda 1 \in (V^* \oplus \mathbb{R})^+ \cap -(V^* \oplus \mathbb{R})^+$. Then we have $f(v) + \lambda = 0$ for all $0 \leqslant v \leqslant e_V$. Taking $v = 0$, we see that $\lambda = 0$ and $f = 0$, thus $(V^* \oplus \mathbb{R})^+ \cap -(V^* \oplus \mathbb{R})^+ = \{0\}$. Since

$$(\|f\| + |\lambda|)1 + (f + \lambda 1) \in (V^* \oplus \mathbb{R})^+,$$

1 is an order unit. Suppose that $f + \lambda 1 + \varepsilon 1$ belongs to $(V^* \oplus \mathbb{R})^+$ for any $\varepsilon > 0$. If $0 \leqslant x \leqslant e_V$, then we have $f(x) + \lambda + \varepsilon \geqslant 0$ for any $\varepsilon > 0$. It follows that $f(x) + \lambda \geqslant 0$, that is, $f + \lambda 1 \in (V^* \oplus \mathbb{R})^+$. □

Note that $f + \lambda 1 \in (V^* \oplus \mathbb{R})^+$ implies $\lambda \geqslant 0$ and the inclusion $\iota : V^* \hookrightarrow V^* \oplus \mathbb{R}$ is an order embedding.

Lemma 3.3. *Suppose that $T : V \to \ell^\infty(\Omega)$ is a unital linear map for a real AOU space V and a discrete set Ω. Then there exists a unital positive map $S : V \to \ell^\infty(\Omega)$ such that $\|T - S\| \leqslant \|T\| - 1$.*

Proof. Let $V \subset C(K)$ for a compact set K [Ka]. By Hahn-Banach theorem, $T : V \to \ell^\infty(\Omega)$ extends to $\tilde{T} : C(K) \to \ell^\infty(\Omega)$ with $\|T\| = \|\tilde{T}\|$. So, we may assume that $V = C(K)$. A unital linear map $T : C(K) \to \ell^\infty(\Omega)$ can be written as $T = \oplus_{\alpha\in\Omega}\mu_\alpha$ for $\mu_\alpha \in C(K)^*$ with $\mu_\alpha(1_K) = 1$ and $\|\mu_\alpha\| \leqslant \|T\|$. We can regard μ_α as a real measure on the compact set K. Let $\mu_\alpha = \mu_\alpha^+ - \mu_\alpha^-$ be the Jordan decomposition of the real measure μ_α. It follows that

$$\|\mu_\alpha\| = |\mu_\alpha|(1) = \mu_\alpha^+(1) + \mu_\alpha^-(1) \quad \text{and} \quad 1 = \mu_\alpha(1) = \mu_\alpha^+(1) - \mu_\alpha^-(1).$$

We set $\nu_\alpha = \mu_\alpha^+(1)^{-1}\mu_\alpha^+$, which is a probability measure on K. It follows that

$$\begin{aligned} \|\mu_\alpha - \nu_\alpha\| &\leqslant \|\mu_\alpha^-\| + \|\mu_\alpha^+ - \mu_\alpha^+(1)^{-1}\mu_\alpha^+\| \\ &= \mu_\alpha^-(1) + (1 - \mu_\alpha^+(1)^{-1})\mu_\alpha^+(1) \\ &= \|\mu_\alpha\| - 1. \end{aligned}$$

The unital positive map

$$S := (\nu_\alpha)_{\alpha\in\Omega} : V \to \ell^\infty(\Omega)$$

satisfies the claim. □

Although the proof of the following lemma is identical to the proof of [BO, Corollary B.11], we include it for the convenience of the reader.

Lemma 3.4. *Suppose that $T : E \to V$ is a unital linear map for a real AOU space V and a finite dimensional real AOU space E. Then there exists a positive map $S : E \to V$ such that $\|T - S\| \leqslant \dim(E)(\|T\| - 1)$.*

Proof. Let $\iota : V \subset \ell^\infty(\Omega)$ be an inclusion, for example, $\Omega = S(V)$. By Lemma 3.3, there exists a unital positive map $S' : E \to \ell^\infty(\Omega)$ such that $\|\iota \circ T - S'\| \leqslant \|T\| - 1$. We take an Auerbach basis $\{x_i\}_{i=1}^n$ for E, where n denotes the dimension of E. Let $E \subset C(K)$ for a compact set K. The dual functional $\hat{x}_i : E \to \mathbb{R}$ can be extended on $C(K)$ with

$\|\hat{x}_i\| = 1$. The functional $\hat{x}_i : C(K) \to \mathbb{R}$ can be regarded as a real measure on K. We denote the total variation of $\hat{x}_i$ by $|\hat{x}_i|$. We let

$$S = T + \|S' - T\| \sum_{i=1}^{n} |\hat{x}_i|.$$

For $0 \leqslant a \leqslant e_E$, we have

$$\begin{aligned} S(a) &= T(a) + \|S' - T\| \sum_{i=1}^{n} |\hat{x}_i|(a) \\ &\geqslant T(a) + \sum_{i=1}^{n} \hat{x}_i(a)(S'(a) - T(a)) \\ &= S'(a) \\ &\geqslant 0 \end{aligned}$$

because $\sum_{i=1}^{n} \hat{x}_i \otimes (S' - T)$ is a linear map which maps x_i to $S'(x_i) - T(x_i)$ for each $1 \leqslant i \leqslant n$. We also see that

$$\|S - T\| \leqslant \|S' - T\| \sum_{i=1}^{n} \|\hat{x}_i\| = n\|S' - T\| \leqslant n(\|T\| - 1).$$

□

Lemma 3.5. *Suppose that $f : V \to \mathbb{R}$ is a linear functional for a real AOU space V. Then we have $f(x) \geqslant -\varepsilon$ for all $0 \leqslant x \leqslant e_V$ if and only if $\|f\| \leqslant 2\varepsilon + f(e_V)$.*

Proof. $\Rightarrow$) Since $0 \leqslant e_V \pm \|x\|^{-1} x \leqslant 2e_V$, we have

$$-2\varepsilon \leqslant f(e_V) \pm \frac{1}{\|x\|} f(x),$$

equivalently,

$$-(f(e_V) + 2\varepsilon)\|x\| \leqslant f(x) \leqslant (f(e_V) + 2\varepsilon)\|x\|.$$

$\Leftarrow$) For $0 \leqslant x \leqslant e_V$, we have

$$-\frac{1}{2} e_V \leqslant x - \frac{1}{2} e_V \leqslant \frac{1}{2} e_V, \text{thus} \quad \|x - \frac{1}{2} e_V\| \leqslant \frac{1}{2}.$$

By the assumption, we have

$$|f(x - \frac{1}{2} e_V)| \leqslant \varepsilon + \frac{1}{2} f(e_V), \text{thus} \quad f(x) \geqslant -\varepsilon.$$

□

Theorem 3.6. *A real AOU space V is nuclear if and only if there exist nets of unital positive maps $\Phi_\lambda : V \to \ell^\infty_{n_\lambda}$ and $\Psi_\lambda : \ell^\infty_{n_\lambda} \to V$ such that $\Psi_\lambda \circ \Phi_\lambda$ converges to id_V in the point-norm topology.*

$$\begin{tikzcd} V \arrow[rr, "\mathrm{id}_V"] \arrow[dr, "\Phi_\lambda"'] & & V \\ & \ell^\infty_{n_\lambda} \arrow[ur, "\Psi_\lambda"'] & \end{tikzcd}$$

Proof. $\Leftarrow$) First, let us show that ℓ_n^∞ is nuclear. We choose an element $\sum_{k=1}^n e_k \otimes w_k$ in $(\ell_n^\infty \otimes_\varepsilon W)^+$. We have

$$0 \leqslant (e_j' \otimes f)(\sum_{i=1}^n e_i \otimes w_i) = f(w_j)$$

for all $f \in S(W)$ and $1 \leqslant j \leqslant n$. We see that $w_j \in W^+$, thus

$$\sum_{i=1}^n e_i \otimes w_i \in \ell_n^{\infty +} \otimes W^+.$$

Hence, ℓ_n^∞ is nuclear. The map

$$\Psi_\lambda \circ \Phi_\lambda \otimes \mathrm{id}_W : V \otimes_\varepsilon W \to \ell_{n_\lambda}^\infty \otimes_\varepsilon W = \ell_{n_\lambda}^\infty \otimes_\pi W \to V \otimes_\pi W$$

is unital positive. Since $\| \cdot \|_{V \otimes_\pi W}$ is a cross norm, $\Psi_\lambda \circ \Phi_\lambda \otimes \mathrm{id}_W(z)$ converges to z for each $z \in V \otimes W$. It follows that $z \in (V \otimes_\varepsilon W)^+$ implies $z \in (V \otimes_\pi W)^+$.

$\Rightarrow$) We choose a finite subset $\{v_1, \cdots, v_m\} \subset V$ and $\varepsilon > 0$. There exists a finite set of states $\{\varphi_1, \cdots \varphi_n\} \subset S(V)$ such that the unital positive map

$$\Phi := \varphi_1 \oplus \cdots \oplus \varphi_n : \mathrm{span}\{e_V, v_1, \cdots, v_m\} \to \ell_n^\infty$$

is injective and the inequality

$$\|\Phi^{-1}|_{\mathrm{Im}\Phi} : \mathrm{Im}\Phi \to V\| \leqslant 1 + \varepsilon$$

holds. By Lemma 3.4, we obtain nets of unital positive maps $\Phi_\lambda : V \to E_\lambda \subset \ell_{n_\lambda}^\infty$ and positive maps $T_\lambda : E_\lambda \to V$ such that $T_\lambda \circ \Phi_\lambda$ converges to id_V in the point-norm topology.

Since E_λ is finite dimensional, the positive linear map $T_\lambda : E_\lambda \to V$ can be regarded as an element in $E_\lambda^* \otimes V \subset (E_\lambda^* \oplus \mathbb{R}) \otimes V$. We take a state f on $E_\lambda^* \oplus \mathbb{R}$. Since $f|_{E_\lambda^*}$ is a positive linear functional on E_λ^*, it is a weak* limit of the elements in E_λ^+. So, the positive linear map T_λ belongs to $((E_\lambda^* \oplus \mathbb{R}) \otimes_\varepsilon V)^+$. By the nuclearity of V, we have

$$T_\lambda \in ((E_\lambda^* \oplus \mathbb{R}) \otimes_\varepsilon V)^+ = ((E_\lambda^* \oplus \mathbb{R}) \otimes_\pi V)^+,$$

that is, $T_\lambda + \varepsilon 1 \otimes e_V \in (E_\lambda^* \oplus \mathbb{R})^+ \otimes V^+$ for any $\varepsilon > 0$. We write

$$T_\lambda + \varepsilon 1 \otimes e_V = \sum_{i=1}^n (f_i + \lambda_i 1) \otimes v_i$$

where $f_i + \lambda_i 1 \in (E_\lambda^* \oplus \mathbb{R})^+$ and $v_i \in V^+$. Then we have

$$T_\lambda(x) = \sum_{i=1}^n f_i(x) v_i + \sum_{i=1}^n \lambda_i v_i - \varepsilon e_V$$

for all $x \in E_\lambda$. Taking $x = 0$, we see that

$$T_\lambda(x) = \sum_{i=1}^n f_i(x) v_i \quad \text{and} \quad \varepsilon e_V = \sum_{i=1}^n \lambda_i v_i.$$

If $0 \leqslant x \leqslant e_{E_\lambda}$, then we have $f_i(x) \geqslant -\lambda_i$ for each i. By Lemma 3.5 and Hahn-Banach theorem, $f_i : E_\lambda \to \mathbb{R}$ extends to $\tilde{f}_i : \ell_{n_\lambda}^\infty \to \mathbb{R}$ such that $\tilde{f}_i(x) \geqslant -\lambda_i$ for all $0 \leqslant x \leqslant 1_{\ell_{n_\lambda}^\infty}$. We can regard $\tilde{f}_i$ as a finite real sequence with length n_λ. We set

$$\tilde{T}_\lambda = \sum_{i=1}^n \tilde{f}_i \otimes v_i \quad \text{and} \quad R_\lambda = \sum_{i=1}^n \tilde{f}_i^+ \otimes v_i.$$

Then $\tilde{T}_\lambda : \ell^\infty_{n_\lambda} \to V$ is an extension of $T_\lambda : E_\lambda \to V$ and R_λ is a positive linear map from $\ell^\infty_{n_\lambda}$ to V.

$$\begin{tikzcd}
V \arrow[rr, "\mathrm{id}_V"] \arrow[dr, "\Phi_\lambda"'] & & V \\
& E_{n_\lambda} \arrow[ur, "T_\lambda"'] \arrow[r, hook] & \ell^\infty_{n_\lambda} \arrow[u, "\tilde{T}_\lambda, R_\lambda"']
\end{tikzcd}$$

For a contractive element x in $\ell^\infty_{n_\lambda}$, we have

$$\begin{aligned}
\|\tilde{T}_\lambda(x) - R_\lambda(x)\| &= \|\sum_{i=1}^n \tilde{f}_i(x)v_i - \sum_{i=1}^n \tilde{f}_i^+(x)v_i\| \\
&= \|\sum_{i=1}^n \tilde{f}_i^-(x)v_i\| \\
&\leqslant \|\sum_{i=1}^n \tilde{f}_i^-(1_{\ell^\infty_{n_\lambda}})v_i\| \\
&\leqslant \|\sum_{i=1}^n \lambda_i v_i\| \\
&= \varepsilon.
\end{aligned}$$

Hence, we can take nets of unital positive maps $\Phi_\lambda : V \to \ell^\infty_{n_\lambda}$ and positive maps $T_\lambda : \ell^\infty_{n_\lambda} \to V$ such that $T_\lambda \circ \Phi_\lambda$ converges to id_V in the point norm topology. Since each Φ_λ is unital, $T_\lambda(1_{\ell^\infty_{n_\lambda}})$ converges to e_V. Let us choose a state ω on V and set

$$\Psi_\lambda(x) = \frac{1}{\|T_\lambda\|}T_\lambda(x) + \omega(x)(e_V - \frac{1}{\|T_\lambda\|}T_\lambda(1)).$$

Then $\Psi_\lambda : \ell^\infty_{n_\lambda} \to V$ is a unital positive map such that $\Psi_\lambda \circ \Phi_\lambda$ converges to id_V in the point-norm topology. □

Corollary 3.7. *(1) For a compact set K, the real continuous function algebra $C(K)$ is a nuclear real AOU space.*

(2) For real AOU spaces $W_1 \subset W_2$ and a finite dimensional nuclear real AOU space V, a unital positive map $T : W_1 \to V$ extends to a unital positive map $\tilde{T} : W_2 \to V$.

(3) A finite dimensional nuclear real AOU space V is isometric to $\ell^\infty_{\dim V}$.

(4) The space spanned by $\{1, t_1, \cdots, t_n\}$ in $C([-1,1]^n)$ is not nuclear for $n \geqslant 2$.

Proof. (1) There exist nets of unital positive maps $\Phi_\lambda : C(K) \to \ell^\infty_{n_\lambda}$ and $\Psi_\lambda : \ell^\infty_{n_\lambda} \to C(K)$ such that $\Psi_\lambda \circ \Phi_\lambda$ converges to $\mathrm{id}_{C(K)}$ in the point-norm topology [L, Theorem 2.3.7].

(2) Suppose that the identity map $\mathrm{id}_V : V \to V$ factorizes into unital positive maps $\Phi_\lambda : V \to \ell^\infty_{n_\lambda}$ and $\Psi_\lambda : \ell^\infty_{n_\lambda} \to V$ approximately. The unital positive map $\Phi_\lambda \circ T : W_1 \to \ell^\infty_{n_\lambda}$ extends to a unital positive map $\tilde{T}_\lambda : W_2 \to \ell^\infty_{n_\lambda}$. Let $\tilde{T} : W_2 \to V$ be the point-weak* cluster point of the set $\{\Psi_\lambda \circ \tilde{T}_\lambda\}$.

(3) By the same manner as the above (2), we can show that a finite dimensional nuclear real AOU space is an injective Banach space.

(4) The space spanned by $\{1, t_1, \cdots t_n\}$ in $C([-1,1]^n)$ is isometric to ℓ^1_{n+1} □

4. The complex case

In this section, we consider the tensor products of (complex) AOU spaces. Because the most proofs are similar to those in section 2 and 3, the similar proof will be omitted and the only additional proof will be given if it is necessary.

For $*$-vector spaces V and W, the involution on $V \otimes W$ is given by

$$(v \otimes w)^* = v^* \otimes w^*$$

for $v \in V$ and $w \in W$.

Definition 4.1. Suppose that (V, V^+, e_V) and (W, W^+, e_W) are AOU spaces.

(1) we define *an injective tensor product* $V \otimes_\varepsilon W$ as $(V \otimes W, (V \otimes_\varepsilon W)^+, e_V \otimes e_W)$ where $(V \otimes_\varepsilon W)^+ = \{z \in V \otimes W : (f \otimes g)(z) \geqslant 0 \text{ for all } f \in S(V), g \in S(W)\}$.

(2) we define *a projective tensor product* $V \otimes_\pi W$ as $(V \otimes W, (V \otimes_\pi W)^+, e_V \otimes e_W)$ where $(V \otimes_\pi W)^+ = \{z \in V \otimes W : z + \varepsilon e_V \otimes e_W \in V^+ \otimes W^+ \text{ for all } \varepsilon > 0\}$.

Lemma 4.2. *For $*$-vector spaces V and W, we have*

$$V_h \otimes W_h = (V \otimes W)_h.$$

Proof. Let $z = \sum_{k=1}^n v_k \otimes w_k \in (V \otimes W)_h$. The conclusion follows from

$$\begin{aligned} z &= \frac{1}{2}\sum_{k=1}^n v_k \otimes w_k + \frac{1}{2}\sum_{k=1}^n v_k^* \otimes w_k^* \\ &= \sum_{k=1}^n (\frac{v_k + v_k^*}{2}) \otimes (\frac{w_k + w_k^*}{2}) - \sum_{k=1}^n (\frac{v_k - v_k^*}{2i}) \otimes (\frac{w_k - w_k^*}{2i}) \\ &\in V_h \otimes W_h. \end{aligned}$$

□

For an ordered $*$-space V and a real functional $f : V_h \to \mathbb{R}$, the complex functional $\tilde{f} : V \to \mathbb{C}$ is defined by

$$\tilde{f}(v) = f(\frac{v + v^*}{2}) + if(\frac{v - v^*}{2i})$$

[PT, Definition 3.9]. Then $f : V_h \to \mathbb{R}$ is a state if and only if $\tilde{f} : V \to \mathbb{C}$ is a state [PT, Proposition 3.10] and every state on V is realized in this form [PT, Proposition 3.11].

Theorem 4.3. *Suppose that (V, V^+, e_V) and (W, W^+, e_W) are AOU spaces. Then*

(1) *the injective tensor product $V \otimes_\varepsilon W$ is an AOU space.*

(2) *the projective tensor product $V \otimes_\pi W$ is an AOU space.*

(3) *the minimal order norm induced by the injective tensor product is a cross norm with respect to the minimal order norms of V and W.*

(4) *the maximal order norm induced by the projective tensor product is a subcross norm with respect to the maximal order norms of V and W.*

Proof. (1) Let us show that $(V \otimes_\varepsilon W)^+ \subset (V \otimes W)_h$. Let $z = \sum_{k=1}^n v_k \otimes w_k \in (V \otimes_\varepsilon W)^+$. Since $v = \frac{v+v^*}{2} + i\frac{v-v^*}{2i}$, we may assume that each v_k is hermitian. Considering basis for the real space spanned by $\{v_k\}_{k=1}^n$, we may assume that $\{v_k\}_{k=1}^n$ is linearly independent. For $f \in S(V)$ and $g \in S(W)$, we have

$$\sum_{k=1}^n f(v_k) g(w_k^*) = \sum_{k=1}^n f(v_k)\overline{g(w_k)} = \overline{(f \otimes g)(z)} = (f \otimes g)(z) = \sum_{k=1}^n f(v_k) g(w_k).$$

It follows that

$$f(\sum_{k=1}^{n} g(w_k - w_k^*)v_k) = 0$$

for all $f \in S(V)$. By [PT, Prop 3.12], we have $\sum_{k=1}^{n} g(w_k - w_k^*)v_k = 0$. Since $\{v_k\}_{k=1}^{n}$ is linearly independent, we have $g(w_k - w_k^*) = 0$ for all $g \in S(W)$. By [PT, Prop 3.12] again, we see that w_k is hermitian.

Let $z = \sum_{k=1}^{n} v_k \otimes w_k \in (V \otimes_\varepsilon W)_h$. By Lemma 4.2, the hermitian element z can be written as $z = \sum_{k=1}^{n} v_k \otimes w_k$ for $v_k \in V_h$ and $w_k \in W_h$. By Theorem 2.2, we see that $\sum_{k=1}^{n} \|v_k\| \|w_k\| e_V \otimes e_W \pm z$ belongs to $(V \otimes_\varepsilon W)^+$.

(3) We denote by $\| \cdot \|_{V \otimes_\varepsilon W, m}$ the minimal order norm induced by the injective tensor product $V \otimes_\varepsilon W$. For $v \in V$ and $w \in W$, we have

$$\|v\|_m \|w\|_m = \sup\{|(f \otimes g)(v \otimes w)| : f \in S(V), g \in S(W)\} \leqslant \|v \otimes w\|_{V \otimes_\varepsilon W, m}.$$

For a state F on $V \otimes_\varepsilon W$, we let

$$F(v \otimes w) = e^{i\theta} |F(v \otimes w)| \quad \text{and} \quad u = e^{-i\theta} v.$$

It follows that

$$\begin{aligned}
&|F(v \otimes w)| \\
=&F(u \otimes w) \\
=&F(\frac{1}{2} u \otimes w + \frac{1}{2} u^* \otimes w^*) \\
=&F((\frac{u + u^*}{2}) \otimes (\frac{w + w^*}{2}) - (\frac{u - u^*}{2i}) \otimes (\frac{w - w^*}{2i})) \\
\leqslant&\|(\frac{u + u^*}{2}) \otimes (\frac{w + w^*}{2}) - (\frac{u - u^*}{2i}) \otimes (\frac{w - w^*}{2i})\|_{V_h \otimes_\varepsilon W_h} \\
=&\sup\{|f(\frac{u + u^*}{2}) g(\frac{w + w^*}{2}) - f(\frac{u - u^*}{2i}) g(\frac{w - w^*}{2i})| : f \in S(V_h), g \in S(W_h)\} \\
\leqslant&\sup\{|(f(\frac{u + u^*}{2}) + if(\frac{u - u^*}{2i}))(g(\frac{w + w^*}{2}) + ig(\frac{w - w^*}{2i}))| : f \in S(V_h), g \in S(W_h)\} \\
=&\sup\{|\tilde{f}(u) \tilde{g}(w)| : f \in S(V_h), g \in S(W_h)\} \\
=&\|u\|_m \|w\|_m \\
=&\|v\|_m \|w\|_m.
\end{aligned}$$

(4) We denote by $\| \cdot \|_{V \otimes_\pi W, M}$ the maximal order norm induced by the projective tensor product $V \otimes_\pi W$. For $v \in V$ and $w \in W$, we write

$$v = \sum_{k=1}^{m} \lambda_k v_k \quad \text{and} \quad w = \sum_{l=1}^{n} \mu_l w_l$$

for $\lambda_k, \mu_l \in \mathbb{C}$ and $v_k \in V_h, w_l \in W_h$. Then we have

$$v \otimes w = \sum_{\substack{1 \leqslant k \leqslant m \\ 1 \leqslant l \leqslant n}} \lambda_k \mu_l \; v_k \otimes w_l$$

and
$$(\sum_{k=1}^{m}|\lambda_k|\|v_k\|)(\sum_{l=1}^{n}|\mu_l|\|w_l\|)=\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}}|\lambda_k\mu_l|\|v_k\otimes w_l\|_{V_h\otimes_\pi W_h}.$$
It follows that
$$\|v\otimes w\|_{V\otimes_\pi W,M}\leqslant\|v\|_M\|w\|_M.$$
□

For the definitions of OMIN and OMAX, we refer to [PTT].

Proposition 4.4. *For an AOU space V, we have*
$$M_n(\mathrm{OMIN}(V))^+=(\mathbb{M}_n\otimes_\varepsilon V)^+\quad\textit{and}\quad M_n(\mathrm{OMAX}(V))^+=(\mathbb{M}_n\otimes_\pi V)^+.$$

Proof. For $f\in S(\mathbb{M}_n)$ and $g\in S(V)$, we have
$$f([g(v_{ij})]_{(i,j)})=f(\sum_{i,j=1}^{n}g(v_{ij})e_{ij})=\sum_{i,j=1}^{n}f(e_{ij})g(v_{ij})=(f\otimes g)(\sum_{i,j=1}^{n}e_{ij}\otimes v_{ij}).$$
The first identity follows from [PTT, Theorem 3.2]. □

Proposition 4.5. *Suppose that V,W and Z are AOU spaces and $\Phi:V\times W\to Z$ is a bilinear map such that $\Phi(e_V,e_W)=e_Z$ and $\Phi(v,w)\in Z^+$ for all $v\in V^+$ and $w\in W^+$. Then there exists a unique unital positive linear map $\tilde{\Phi}:V\otimes_\pi W\to Z$ such that $\Phi(v,w)=\tilde{\Phi}(v\otimes w)$. This universal property characterizes the projective tensor product $V\otimes_\pi W$ up to unital order isomorphism.*

Proposition 4.6. *Suppose that $S:V_1\to V_2$ and $T:W_1\to W_2$ are unital positive linear maps for AOU spaces V_1,V_2,W_1,W_2. Then*

(1) *$S\otimes T:V_1\otimes_\varepsilon W_1\to V_2\otimes_\varepsilon W_2$ is a unital positive linear map.*
(2) *$S\otimes T:V_1\otimes_\pi W_1\to V_2\otimes_\pi W_2$ is a unital positive linear map.*

Definition 4.7. Suppose that $T:V\to W$ is a unital positive surjective linear map for AOU spaces V and W. We call $T:V\to W$ *an order quotient map* if for any w in W^+ and $\varepsilon>0$, we can take an element v in V so that it satisfies
$$v+\varepsilon e_V\in V^+\quad\text{and}\quad T(v)=w.$$

Proposition 4.8. *Suppose that $T:V\to W$ is a unital positive surjective linear map for AOU spaces V and W. Then $T:V\to W$ is an order quotient map if and only if $\tilde{T}:V/\ker T\to W$ is an order isomorphism.*

Proposition 4.9. *Suppose that $T:V\to V$ is a unital positive linear map for AOU spaces V and W. Then*

(1) *$T:V\to W$ is an order embedding if and only if it is an isometry with respect to the minimal order norms.*
(2) *if $T:V\to W$ is a quotient map with respect to the order norms, then it is an order quotient map.*

Proof. (1) It follows from [PT, Theorem 4.22].
(2) In the proof of Proposition 2.7, we consider the hermitian lifting $\frac{1}{2}(v+v^*)$. □

Theorem 4.10. *(1) For AOU spaces V_1, V_2, W and a unital order embedding $\iota : V_1 \to V_2$, the linear map $\iota \otimes id_W : V_1 \otimes_\varepsilon W \to V_2 \otimes_\varepsilon W$ is a unital order embedding.*

(2) For AOU spaces V_1, V_2, W and an order quotient map $Q : V_1 \to V_2$, the linear map $Q \otimes id_W : V_1 \otimes_\pi W \to V_2 \otimes_\pi W$ is an order quotient map.

Proof. (1) Combining [PT, Corollary 2.15] with [PT, Proposition 3.11], we obtain Hahn-Banach type theorem for a state on AOU space. □

Proposition 4.11. *For AOU spaces V and W, the minimal order norm on the injective tensor product $V \otimes_\varepsilon W$ coincides with the injective tensor norm on $V_m \otimes_\lambda W_m$.*

Definition 4.12. An AOU space V is called *nuclear* if the identity

$$V \otimes_\varepsilon W = V \otimes_\pi W$$

holds for any AOU space W.

Lemma 4.13. *An AOU space V is nuclear if and only if the real AOU space V_h is nuclear.*

Proof. Given an AOU space V, we get the real AOU space V_h. Conversely, given a real AOU space W, we get the AOU space $W^{\mathbb{C}}$ by the complexification $W^{\mathbb{C}} := W \otimes_{\mathbb{R}} \mathbb{C}$. In other words, there are one-to-one correspondences between real AOU spaces and AOU spaces. It is easy to check that

$$(V_h \otimes_\varepsilon W)^+ = (V \otimes_\varepsilon W^{\mathbb{C}})^+ \quad \text{and} \quad (V_h \otimes_\pi W)^+ = (V \otimes_\pi W^{\mathbb{C}})^+$$

for an AOU space V and a real AOU space W. □

Theorem 4.14. *An AOU space V is nuclear if and only if there exist nets of unital positive maps $\Phi_\lambda : V \to \ell^\infty_{n_\lambda}$ and $\Psi_\lambda : \ell^\infty_{n_\lambda} \to V$ such that $\Psi_\lambda \circ \Phi_\lambda$ converges to id_V in the point-norm topology.*

Proof. V is nuclear

$\Leftrightarrow$ V_h is nuclear

$\Leftrightarrow$ there exist nets of unital positive maps $\Phi_\lambda : V_h \to \ell^\infty_{n_\lambda}(\mathbb{R})$ and $\Psi_\lambda : \ell^\infty_{n_\lambda}(\mathbb{R}) \to V_h$ such that $\Psi_\lambda \circ \Phi_\lambda$ converges to id_V in the point-norm topology

$\Leftrightarrow$ there exist nets of unital positive maps $\Phi_\lambda : V \to \ell^\infty_{n_\lambda}(\mathbb{C})$ and $\Psi_\lambda : \ell^\infty_{n_\lambda}(\mathbb{C}) \to V$ such that $\Psi_\lambda \circ \Phi_\lambda$ converges to id_V in the point-norm topology.

□

Corollary 4.15. *(1) For a compact set K, $C(K)$ is a nuclear AOU space.*

(2) For AOU spaces $W_1 \subset W_2$ and a finite dimensional nuclear AOU space V, a unital positive map $T : W_1 \to V$ extends to a unital positive map $\tilde{T} : W_2 \to V$.

(3) A finite dimensional nuclear AOU space V is isometric to $\ell^\infty_{\dim V}$.

(4) The space spanned by $\{1, z_1, \cdots, z_n\}$ in $C(\mathbb{T}^n)$ is not nuclear.

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